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Multi-instanton measure from recursion relations in $N = 2$ supersymmetric Yang-Mills theory

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Abstract: By using the recursion relations found in the framework of $N = 2$ Super Yang-Mills theory with gauge group SU(2), we reconstruct the structure of the instanton moduli space and its volume form for all winding numbers.

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In [1] the entire nonperturbative contribution to the holomorphic part of the Wilsonian effective action was computed for $N = 2$ globally supersymmetric (SUSY) theories with gauge group SU(2), using ansätze dictated by physical intuitions. There are several aspects of the Seiberg-Witten (SW) model [1] which are related to the theory of moduli spaces of Riemann surfaces. In particular, here, we will consider the recursion relations for nonperturbative (instanton) contributions to the $N = 2$ Super Yang-Mills (SYM) effective prepotential [2] and will compare them with the recursion relations for the Weil-Petersson volumes of punctured Riemann spheres. In the Seiberg-Witten model there exists a relation the modulus $u = \langle \mathrm{Tr}\, \phi^2 \rangle$ and the effective prepotential [2] (see also [3]). This allowed to prove the SW conjecture by using the reflection symmetry of vacua [4]. On the other hand, it is rather surprising that, while on one side all the instanton coefficients have been computed in [2], explicit calculations have been performed only in the one and two-instanton background [5, 6, 7], while the above mentioned relation has been shown to hold to all instanton orders [8, 9]. The problem for instanton number $k \geq 3$ seems extremely difficult to solve. Indeed, the ADHM constraint equations become nonlinear and have not been explicitly solved up to now. Moreover, neither the structure of the moduli space, nor the volume form are known. The instanton measure for all winding numbers has been written in [10], but only in an implicit form (i.e. by implementing the bosonic and fermionic ADHM constraints through the use of Dirac delta functions), which in some special cases allows to extract information on the instanton moduli space [11]. However, the mathematical challenging problem of finding the explicit structure of the instanton moduli space for generic winding numbers still remains unsolved. On the other hand, the simple way in which the recursion relations have been derived, strongly suggests that there may be some mechanism which should make the explicit calculations possible. The investigation of such mechanism would provide important information on the structure of the instanton moduli space (of which only the boundary à la Donaldson-Uhlenbeck is known for generic winding number [12, 13, 14]) and of the associated volume form. In particular, even if the integrals seem impossible to compute, (actually, as we stated before we know neither the structure of the space nor the volume form), the existence of recursion relations and the simple way in which they arise, seem to suggest that these integrals could be easy to compute because of some underlying geometrical recursive structure. It has been claimed for some time, but only recently proven [15], that the nonperturbative contributions to u actually can be written as total derivatives, i.e. as pure boundary terms, on the moduli space. If the boundary is composed by moduli spaces of instantons of lower winding number times zero-size instantons moduli spaces, as it happens in the Donaldson-Uhlenbeck compactification, this would immediately provide, in the case of a suitable volume form, a recursion relation.

We will now see how the similar problem one finds in computing the Weil-Petersson (WP) volumes of punctured spheres has been solved thanks to the re-

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cursive structure of the Deligne-Knudsen-Mumford boundary and to the peculiar nature of the WP 2-form. The main analogy we will display, concerns the volume of moduli space of n-punctured Riemann spheres $\Sigma_{0,n} = \widehat{\mathbb{C}}\backslash\{z_1, \ldots, z_n\}$, $n \geq 3$, where $\widehat{\mathbb{C}} \equiv \mathbb{C} \cup \{\infty\}$. Their moduli space is the space of classes of isomorphic $\Sigma_{0,n}$'s, that is

$$\mathcal{M}_{0,n} = \{(z_1, \ldots, z_n) \in \widehat{\mathbb{C}}^n | z_j \neq z_k \text{ for } j \neq k\}/\text{Symm}(n) \times PSL(2, \mathbb{C}) , \qquad (1)$$

where Symm(n) acts by permuting $\{z_1, \ldots, z_n\}$ whereas $PSL(2, \mathbb{C})$ acts as a linear fractional transformation. Using $PSL(2, \mathbb{C})$ symmetry we can recover the "standard normalization": $z_{n-2} = 0$, $z_{n-1} = 1$ and $z_n = \infty$. The classical Liouville tensor or Fuchsian projective connection is

$$T^F(z) = \left\{J_H^{-1}, z\right\} = \varphi_{cl\,zz} - \frac{1}{2}\varphi_{cl\,z}^2 \, . \qquad (2)$$

In the case of the punctured Riemann sphere we have

$$T^F(z) = \sum_{k=1}^{n-1} \left(\frac{1}{2(z - z_k)^2} + \frac{c_k}{z - z_k} \right) , \qquad (3)$$

where the coefficients $c_1, \ldots c_{n-1}$, called *accessory parameters*, satisfy the constraints

$$\sum_{j=1}^{n-1} c_j = 0 \, , \qquad \sum_{j=1}^{n-1} z_j c_j = 1 - \frac{n}{2} \, . \qquad (4)$$

These parameters are defined on the space

$$V^{(n)} = \{(z_1, \ldots, z_{n-3}) \in \mathbb{C}^{n-3} | z_j \neq 0, 1 \quad ; z_j \neq z_k \, , \text{ for } j \neq k\} \, . \qquad (5)$$

Note that

$$\mathcal{M}_{0,n} \cong V^{(n)}/\text{Symm}(n) \, , \qquad (6)$$

where the action of Symm(n) on $V^{(n)}$ is defined by comparing (1) with (6).

Let us now consider the compactification $\overline{V}^{(n)}$ à la Deligne-Knudsen-Mumford [16, 17]. The divisor at the boundary

$$D = \overline{V}^{(n)} \backslash V^{(n)} \, , \qquad (7)$$

decomposes in the sum of divisors $D_1, \ldots, D_{[n/2]-1}$, which are subvarieties of real dimension $2n - 8$. The locus D_k consists of surfaces that split, upon removal of the node, into two Riemann spheres with $k + 2$ and $n - k$ punctures. In particular, D_k consists of $C(k)$ copies of the space $\overline{V}^{(k+2)} \times \overline{V}^{(n-k)}$ where $C(k) = \binom{n}{k+1}$, for $k = 1, \ldots, (n - 3)/2$, n odd. In the case of even n the unique difference is for $k = n/2 - 1$, for which we have $C(n/2 - 1) = \frac{1}{2}\binom{n}{n/2}$. An important property of the divisors D_k's is that their image provides a basis in $H_{2n-8}(\overline{\mathcal{M}}_{0,n},\mathbb{R})$. The Weil-Petersson volume is

$$\mathrm{Vol}_{WP}\left(\mathcal{M}_{0,n}\right)=\frac{1}{(n-3)!}\int_{\overline{\mathcal{M}}_{0,n}}\omega_{WP}^{(n)}{}^{\,n-3}=\frac{1}{(n-3)!}\left[\omega_{WP}^{(n)}\right]^{n-3}\cap\left[\overline{\mathcal{M}}_{0,n}\right]. \quad (8)$$

It has been shown that [17]

$$\mathrm{Vol}_{WP}\left(\mathcal{M}_{0,n}\right)=\frac{1}{n!}\mathrm{Vol}_{WP}\left(V^{(n)}\right)=\frac{\pi^{2(n-3)}V_n}{n!(n-3)!}, \qquad n\geq 4, \quad (9)$$

where $V_n=\pi^{2(3-n)}\left[\omega_{WP}^{(n)}\right]^{n-3}\cap\left[\overline{V}^{(n)}\right]$ satisfies the recursion relations

$$V_3=1, \qquad V_n=\frac{1}{2}\sum_{k=1}^{n-3}\frac{k(n-k-2)}{n-1}\binom{n}{k+1}\binom{n-4}{k-1}V_{k+2}V_{n-k}, \qquad n\geq 4. \quad (10)$$

These recursive relations are a consequence of two basic properties. The first one is the fact that the boundary of the moduli space in the Deligne-Knudsen-Mumford compactification is the union of product of moduli spaces of lower order. The second one is the restriction phenomenon satisfied by the Weil-Petersson 2-form. A property discovered by Wolpert in [18] (see also [19, appendix]). The basic idea is to start with the natural embedding

$$i:\overline{V}^{(m)}\to\overline{V}^{(m)}\times *\to\overline{V}^{(m)}\times\overline{V}^{(n-m+2)}\to\partial\overline{V}^{(n)}\to\overline{V}^{(n)}, \qquad n>m, \quad (11)$$

where $*$ is an arbitrary point in $\overline{V}^{(n-m+2)}$, it follows that [18]

$$\left[\omega_{WP}^{(m)}\right]=i^*\left[\omega_{WP}^{(n)}\right], \qquad n>m. \quad (12)$$

There is a similarity between the above recursion relations for the WP volumes and the recursion relations satisfied by the instanton coefficients. To see this let us recall that in the case of the WP volumes, it has been derived in [19] a nonlinear differential equation satisfied by the generating function for the Weil-Petersson volumes

$$g(x)=\sum_{k=3}^{\infty}a_kx^{k-1}, \quad (13)$$

where

$$a_k=\frac{V_k}{(k-1)((k-3)!)^2}, \qquad k\geq 3, \quad (14)$$

so that (10) assumes the simple form

$$a_3=1/2, \qquad a_n=\frac{1}{2}\frac{n(n-2)}{(n-1)(n-3)}\sum_{k=1}^{n-3}a_{k+2}a_{n-k}, \qquad n\geq 4. \quad (15)$$

One can check that (15) implies that the function g satisfies the differential equation [19]

$$x(x-g)g'' = xg'^2 + (x-g)g' . \tag{16}$$

Remarkably, it has been shown in [20], that this nonlinear differential equation is essentially the inverse of a linear differential (Bessel) equation. More precisely, defining $g = x^2 \partial_x x^{-1} h$, one has that (16) implies

$$xh'' - h' = (xh' - h)h'' . \tag{17}$$

Differentiating (17) we get

$$yy'' = xy^3 , \tag{18}$$

where $y = h'$. Then, interchanging the rôles of x and y, (18) transforms into the Bessel equation

$$y\frac{d^2x}{dy^2} + x = 0 . \tag{19}$$

It has been suggested in [20] that the appearance of such a linear differential equation may be related to the "mirror phenomenon".

The above structure is reminiscent of the above derived in Seiberg-Witten theory. In particular, in the case of WP volumes one starts evaluating the recursion relations by means of the Deligne-Knudsen-Mumford compactification and the Wolpert restriction phenomenon [17], then derives the associated nonlinear ODE [19] and end to a linear ODE [20] which is obtained by essentially inverting it. In the Seiberg-Witten model, one starts by observing that the $a^D(u)$ and $a(u)$ moduli satisfy a linear ODE [21], inverts the equation to obtain a nonlinear one satisfied by $u(a)$ then finds recursion relations for the coefficients of the expansion of $u(a)$ [2]. The final point stems from the observation that u and $\mathcal{F}$ are related in a simple way which allows one to consider the derived recursion relation as a relation for the instanton contributions to the preportential $\mathcal{F}$. The above similarity suggests to reconstruct the instanton moduli space and its measure starting from the recursion relations [2]

$$\begin{aligned}\mathcal{G}_{n+1} = \frac{1}{8\mathcal{G}_0^2(n+1)^2}\Bigg[&(2n-1)(4n-1)\mathcal{G}_n + 2\mathcal{G}_0\sum_{k=0}^{n-1} c_{k,n}\mathcal{G}_{n-k}\mathcal{G}_{k+1} - \\ &- 2\sum_{j=0}^{n-1}\sum_{k=0}^{j+1} d_{j,k,n}\mathcal{G}_{n-j}\mathcal{G}_{j+1-k}\mathcal{G}_k\Bigg] ,\end{aligned} \tag{20}$$

where $n \geq 0$, $\mathcal{G}_0 = 1/2$ and

$$c_{k,n} = 2k(n-k-1)+n-1 , \qquad d_{j,k,n} = [2(n-j)-1][2n-3j-1+2k(j-k+1)] . \tag{21}$$

It is still possible to rewrite some apparently cubic terms in the third term on the r.h.s. as quadratic ones and absorb them in the second term on the r.h.s. of (20), obtaining thus

$$\mathcal{G}_{n+1} = \frac{1}{2(n+1)^2}\Bigg[(2n-1)(4n-1)\mathcal{G}_n + \sum_{k=0}^{n-1} b_{k,n}\mathcal{G}_{n-k}\mathcal{G}_{k+1} - \\ - 2\sum_{j=1}^{n-1}\sum_{k=1}^{j} d_{j,k,n}\mathcal{G}_{n-j}\mathcal{G}_{j+1-k}\mathcal{G}_k\Bigg], \qquad (22)$$

where $b_{k,n} = c_{k,n} - 2d_{k,0,n}$ and we have exploited the fact that $d_{k,0,n} = d_{k,k+1,n}$. Let us now consider the volume $\mathcal{G}_n$ of the moduli space of an instanton configuration of winding number n. In order to reproduce the recursion relation, we assume that $\mathcal{G}_n$ can be written as

$$\mathcal{G}_n = \int_{\overline{V}_I^{(n)}} \bigwedge_{k=1}^{X(n)} \omega_I^{(n)} = \left[\omega_I^{(n)}\right]^{X(n)} \cap \left[\overline{V}_I^{(n)}\right], \qquad (23)$$

where $\cap$ is the topological cup product, $\omega_I^{(n)}$ is the natural 2-form defined on the n-instanton moduli space and $\overline{V}_I^{(n)}$ is a suitable compactification of $V_I^{(n)}$, which we will make explicit later. The function $X(n)$, representing the complex dimension of $\overline{V}_I^{(n)}$, will be fixed later. It is possible to recast (23) in the form

$$\begin{aligned} \mathcal{G}_{n+1} &= \left[\omega_I^{(n+1)}\right]^{X(n+1)-1} \cap \left(\left[\omega_I^{(n+1)}\right] \cap \left[\overline{V}_I^{(n+1)}\right]\right) \\ &= \left[\omega_I^{(n+1)}\right]^{X(n+1)-1} \cap \left[\mathcal{D}_\omega^{(n+1)} \cdot \overline{V}_I^{(n+1)}\right] \\ &= \left[\omega_I^{(n+1)}\right]^{X(n+1)-1} \cap \left[\mathcal{D}_\omega^{(n+1)}\right], \end{aligned} \qquad (24)$$

where $\cdot$ denotes the topological intersection and $\mathcal{D}_\omega^{(n+1)}$ is the $[2X(n+1)-2]$-cycle Poincaré dual to the "instanton" class $[\omega_I^{(n+1)}]$. The divisor at the boundary

$$\mathcal{D}^{(n+1)} = \overline{V}_I^{(n+1)} / V_I^{(n+1)}, \qquad (25)$$

decomposes in the sum of divisors $\mathcal{D}_{1,j}$, $\mathcal{D}_{2,j,k}$ and $\mathcal{D}_{3,n}$. In order to make contact with the recursion relation for the $\mathcal{G}_n$'s, we set

$$\begin{aligned} \mathcal{D}_{1,j} &= c_{n,j}^{(1)}\overline{V}_I^{(n-j)} \times \overline{V}_I^{(j+1)}, \\ \mathcal{D}_{2,j,k} &= c_{n,j,k}^{(2)}\overline{V}_I^{(n-j)} \times \overline{V}_I^{(j+1-k)} \times \overline{V}_I^{(k)} \times \overline{V}_I^{(1)}, \\ \mathcal{D}_{3,n} &= c_n^{(3)}\overline{V}_I^{(n)} \times \overline{V}_I^{(1)}. \end{aligned} \qquad (26)$$

Let us now expand $\mathcal{D}_\omega^{(n+1)}$ in terms of the divisors at the boundary of the moduli space, namely

$$\mathcal{D}_\omega^{(n+1)} = \sum_{j=0}^{n-1} d_{n,j}^{(1)}\mathcal{D}_{1,j} + \sum_{j=1}^{n-1}\sum_{k=1}^{j} d_{n,j,k}^{(2)}\mathcal{D}_{2,j,k} + d_n^{(3)}\mathcal{D}_{3,n}. \qquad (27)$$

One can see that consistency requirements on the outlined procedure uniquely determine $X(n)$ to be

$$X(n) = 2n - 1\,. \tag{28}$$

Let us consider the following natural embedding

$$i : \overline{V}_I^{(m)} \to \overline{V}_I^{(m)} \times * \to \overline{V}_I^{(m)} \times \overline{V}_I^{(n-m)} \to \partial \overline{V}_I^{(n)} \to \overline{V}_I^{(n)}\,, \qquad n > m\,, \tag{29}$$

where $*$ is an arbitrary point in $\overline{V}_I^{(n-m)}$. We now impose the following constraint

$$\left[\omega_I^{(m)}\right] = i^* \left[\omega_I^{(n)}\right], \qquad n > m\,. \tag{30}$$

Let us elaborate the three terms on the r.h.s. of (24): the first term is

$$\begin{aligned}
&\left[\omega_I^{(n+1)}\right]^{2(n+1)-2} \cap \left[\overline{V}_I^{(n-j)} \times \overline{V}_I^{(j+1)}\right] = \\
&= \left[\omega_I^{(n-j)} + \omega_I^{(j+1)}\right]^{2(n+1)-2} \cap \left[\overline{V}_I^{(n-j)} \times \overline{V}_I^{(j+1)}\right] \\
&= \binom{2(n+1)-2}{2(n-j)-1} \left(\left[\omega_I^{(n-j)}\right]^{2(n-j)-1} \cap \left[\overline{V}_I^{(n-j)}\right]\right)\left(\left[\omega_I^{(j+1)}\right]^{2(j+1)-1} \cap \left[\overline{V}_I^{(j+1)}\right]\right) \\
&= \binom{2(n+1)-2}{2(n-j)-1} \mathcal{G}_{n-j}\mathcal{G}_{j+1}\,.
\end{aligned} \tag{31}$$

The second term has the form

$$\begin{aligned}
&\left[\omega_I^{(n+1)}\right]^{2(n+1)-2} \cap \left[\overline{V}_I^{(n-j)} \times \overline{V}_I^{(j+1-k)} \times \overline{V}_I^{(k)} \times \overline{V}_I^{(1)}\right] = \\
&= \left[\omega_I^{(n-j)} + \omega_I^{(j+1-k)} + \omega_I^{(k)} + \omega_I^{k}(1)\right]^{2(n+1)-2} \cap \left[\overline{V}_I^{(n-j)} \times \overline{V}_I^{(j+1-k)} \times \overline{V}_I^{(k)} \times \overline{V}_I^{(1)}\right] \\
&= 2k \binom{2(n+1)-2}{2k} \binom{2(n+1)-2-2k}{2(n-j)-1} \left(\left[\omega_I^{(n-j)}\right]^{2(n-j)-1} \cap \left[\overline{V}_I^{(n-j)}\right] \times \right. \\
&\quad \left. \times \left(\left[\omega_I^{(j+1-k)}\right]^{2(j+1-k)-1} \cap \left[\overline{V}_I^{(j+1-k)}\right]\right)\left(\left[\omega_I^{(k)}\right]^{2k} \cap \left[\overline{V}_I^{(k)}\right]\right)\left(\left[\omega_I^{(1)}\right] \cap \left[\overline{V}_I^{(1)}\right]\right)\right. \\
&= \frac{k}{2} \binom{2(n+1)-2}{2k} \binom{2(n+1)-2-2k}{2(n-j)-1} \mathcal{G}_{n-j}\mathcal{G}_{j+1-k}\mathcal{G}_k\,,
\end{aligned} \tag{32}$$

where we used the fact that $\mathcal{G}_1 = 1/4$. Finally, the last term is

$$\left[\omega_I^{(n+1)}\right]^{2(n+1)-2} \cap \left[\overline{V}_I^{(n)} \times \overline{V}_I^{(1)}\right] = \frac{n}{2}\mathcal{G}_n\,. \tag{33}$$

In this way we can recast the recursion relations as

$$\begin{aligned}
\mathcal{G}_{n+1} = {} & \sum_{k=0}^{n-1} \binom{2n}{2(n-k)-1} d^{(1)}_{n,k} c^{(1)}_{n,k} \mathcal{G}_{n-k}\mathcal{G}_{k+1} + \sum_{j=1}^{n-1}\sum_{k=1}^{j} \frac{k}{2}\binom{2n}{2k} \times \\
& \times \binom{2(n-k)}{2(n-j)-1} d^{(2)}_{n,j,k} c^{(2)}_{n,j,k} \mathcal{G}_{n-j}\mathcal{G}_{j+1-k}\mathcal{G}_k + \frac{n}{2} d^{(3)}_n c^{(3)}_n \mathcal{G}_n\,,
\end{aligned} \tag{34}$$

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which can be straightforwardly compared to (22) and gives

$$
\begin{aligned}
&d^{(1)}_{n,k}c^{(1)}_{n,k}\begin{pmatrix}2n\\ 2(n-k)-1\end{pmatrix}=\frac{b_{k,n}}{2(n+1)^2}\,,\\
&d^{(2)}_{n,j,k}c^{(2)}_{n,j,k}\begin{pmatrix}2n\\ 2k\end{pmatrix}\begin{pmatrix}2(n-k)\\ 2(n-j)-1\end{pmatrix}=-\frac{2d_{j,k,n}}{k(n+1)^2}\,,\\
&d^{(3)}_{n}c^{(3)}_{n}=\frac{(2n-1)(4n-1)}{n(n+1)^2}\,.
\end{aligned}
\tag{35}
$$

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